A Laser Limit Cycle – An Analytical Study

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The limit cycle behaviour of the intensity and of the polarization of a monochromatic two-polarization-mode laser with an intensity coupling asymmetry and a linear phase anisotropy is studied analytically. In a previous paper we found such limit cycle oscillations numerically. Explicit formulae are derived by perturbation methods for the case of small intensity coupling asymmetry and for the case of large pumping, respectively, which describe the dependence on the laser parameters of the limit cycle period as well as of its amplitude.

1. Introduction

Great interest has been shown in the electric field polarization behaviour of gas lasers [1–7]. Recent experiments have led to the observation of polarization instabilities and chaos [8]. In a previous article [7] we have formulated the polarization dynamics of a monochromatic two-polarization-mode laser with symmetry breaking by linear phase anisotropy in cooperation with intensity coupling asymmetry, resorting in particular to Van Haeringen's classical paper [1]. Using the circular polarization unit vectors (e_+, e_-) we have, for the slowly varying vector amplitude of the lasing electric field $E(t) \propto u_1(t) e_+ + e^{2i\theta_2 - i\pi/2} u_2(t) e_-$, the following equations of motion in their canonical dimensionless form

$$\dot{u}_1 = p u_1 + \phi u_2 \\ - [(1 - \Delta\beta)|u_1|^2 + (1 + \Delta\beta)|u_2|^2] u_1, \quad (1\,a)$$

$$\dot{u}_2 = p u_2 - \phi u_1 \\ - [(1 + \Delta\beta)|u_1|^2 + (1 - \Delta\beta)|u_2|^2] u_2. \quad (1\,b)$$

Here, the dimensionless parameters p, $\Delta\beta$, and ϕ characterize the pump rate, the intensity coupling asymmetry, and the linear phase anisotropy, respectively. The physical range of $\Delta\beta$ is $-1 < \Delta\beta < 1$ (cf. [7]). θ_2 indicates the direction of the main axes of the linear phase loss tensor. The importance of the linear phase anisotropy for the polarization azimuth has been

pointed out earlier by Van Haeringen [9, 10]. Here we study its influence on the polarization ellipticity. Introducing the useful bilinear variables (cf. [7]).

$$I = |u_1|^2 + |u_2|^2, \quad (2\,a)$$
$$I_- = |u_1|^2 - |u_2|^2, \quad (2\,b)$$
$$y = 2\,\text{Re}(u_1^* u_2), \quad (2\,c)$$
$$z = 2\,\text{Im}(u_1^* u_2), \quad (2\,d)$$

the dynamics is reduced from third order to second order nonlinearity

$$\dot{I} = 2(p - I) I + 2\Delta\beta\, I_-^2, \quad (3)$$
$$\dot{I}_- = 2(p - I) I_- + 2\Delta\beta\, I\, I_- + 2\phi\, y, \quad (4)$$
$$\dot{y} = 2(p - I) y - 2\phi\, I_-, \quad (5)$$
$$\dot{z} = 2(p - I) z. \quad (6)$$

In addition there is the constraint

$$I = \sqrt{I_-^2 + y^2 + z^2}. \quad (7)$$

Note that I, I_-, and y have themselves a closed 3-variable dynamics.

As we obtained in [7], the stable solutions of the above equations in the lasing state $p > 0$ are as follows. Outside the parameter range

$$0 < \Delta\beta < \Delta\beta_c = \frac{2}{\sqrt{1 + (p/\phi)^2} + 1}, \quad (8)$$

we have stable steady state solutions and, more specifically, for $\Delta\beta < 0$ there is a symmetric output steady state, i.e., $I_- = 0$, also $I = p$, $y = 0$, the polarization of the total electric field E is linear. For $\Delta\beta > \Delta\beta_c$ there is an asymmetric output steady state with $|I_-| > I/\sqrt{2}$; this is, furthermore, a bistable state because I_- can

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either be positive or negative; E is almost circularly polarized.

In the parameter window (8) we observe, by numerical integration of (3)–(5), a limit cycle solution. Linear stability analysis of the steady states near the parameter window indicates that at $\Delta\beta = 0$ and $\Delta\beta = \Delta\beta_c$ there are two different kinds of Hopf bifurcations. It is the goal of our present paper to give a detailed analytical study of this interesting behaviour. First the transitions to the time dependent state from both sides of the parameter window (8) when varying $\Delta\beta$ are elucidated (Sect. 2). Using frequency-renormalized perturbation methods we then derive explicit, though approximate, formulae for small $\Delta\beta$ (Sect. 3) and for large p (Sect. 4), which describe the dependence of the limit cycle period and its intensity span as functions of the parameters p, $\Delta\beta$, and ϕ. Our main results are (31) and (51), which can be tested experimentally. In the case of small $\Delta\beta$ the deviation of the limit cycle period from its value for symmetric intensity coupling $\Delta\beta = 0$ (which is $\pi/(2\,\phi)$) increases by a factor proportional to $(p\,\Delta\beta/\phi)^2$ while the intensity oscillation spans the range $I_{max} - I_{min} \approx p\,\Delta\beta$; in the case of large pumping and near the transition point $\Delta\beta_c$, the oscillation is highly anharmonic and $I_{max} - I_{min} \approx 2\,\phi$. The oscillation period exhibits slowing down with the scaling $(\Delta\beta_c - \Delta\beta)^{-1/2}$ towards the transition point $\Delta\beta_c$. Finally some conclusions are offered in Section 5.

2. Transition to the Limit Cycle

The emergence of time dependent solutions can be directly demonstrated by numerical integration of the dynamical equations, see Figs. 1 and 2 for examples. Qualitatively we see that in contrast to the steady state cases, where one of the two variables I_- and y is zero or very small, all three variables can have an appreciable magnitude.

A nonzero ϕ with appropriate $\Delta\beta$ is crucial for the emergence of the time dependent solutions. Time varying behaviour still exists at $\Delta\beta = 0$, although the solution here depends on the initial conditions (see below), while at $\Delta\beta = \Delta\beta_c$ it matches the asymmetric output steady state. These observations demonstrate different types of Hopf bifurcations at the two transition points.

If $\Delta\beta > 0$, the symmetric output state $I_- = 0$ exists but is unstable from the results of linear stability analysis in [7]. We now conclude from (7) and in accordance with (3)–(5), $0 < |I_-| < I$, $2(p-I)\,I < \dot{I} < 2[p-(1-\Delta\beta)\,I]\,I$.
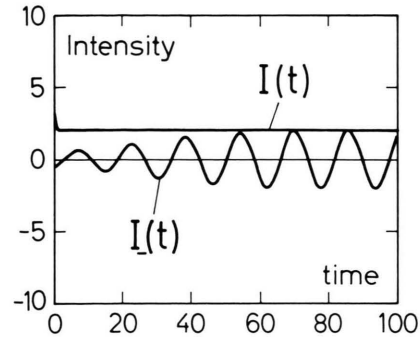

Fig. 1. The intensities $I(t)$ and $I_-(t)$ as functions of time t obtained by numerical integration of (3)–(5). Here $p = 2.0$, $\phi = 0.2$, and $\Delta\beta = 0.02$. Note the transients before entering the periodic state.
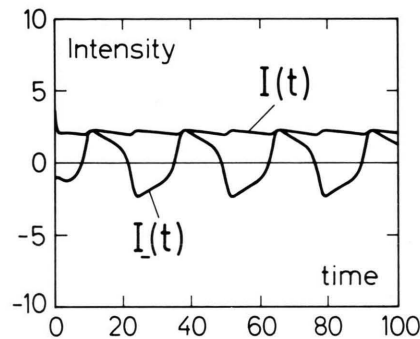

Fig. 2. The same as in Fig. 1 except $\Delta\beta = 0.15$, which is near the transition point $\Delta\beta_c \approx 0.18$. Note that the transients in this case are shorter.

The latter inequality gives an estimate of I,

$$p < I(t) < \frac{p}{1-\Delta\beta}, \qquad \text{when } \Delta\beta > 0. \tag{9}$$

But once $I(t) > p$ is guaranteed, (6) implies $z = 2\,\mathrm{Im}(u_1^* u_2) \to 0$. This means an absolute phase locking between u_1 and u_2, which has already been identified for the asymmetric output steady state in [7].

The approach of z to zero has the consequence (from (7))

$$I = \sqrt{I_-^2 + y^2}. \tag{10}$$

Thus in the time dependent solution range there can be no chaos, because there are effectively only two variables. Instead, there must be a limit cycle.

The above results show that a phase transition happens at $\Delta\beta = 0$ when $\Delta\beta$ goes from negative to positive values, namely,

$$\sqrt{I^2 + I_-^2 + y^2} = \begin{cases} p, & \Delta\beta < 0, \\ 0, & \Delta\beta > 0. \end{cases} \tag{11}$$

$\Delta\beta < 0$ favors a symmetric steady state, while $\Delta\beta > 0$ implies states with symmetry breaking ($I_- \neq 0$), which can be either an asymmetric output steady state ($\Delta\beta \geq \Delta\beta_c$) or a time dependent state ($0 < \Delta\beta < \Delta\beta_c$).

How can the abrupt change expressed by (11) take place? This becomes understandable if one realizes that $\Delta\beta = 0$ is just a case of neutral stability in which the solutions can be exactly evaluated and show dependence on the initial conditions even for $t \to \infty$. Namely, if $\Delta\beta = 0$,

$$I(t) = \frac{p}{1 + [p/I(0) - 1]\, e^{-2pt}} \to p, \qquad (12\,a)$$

$$\begin{aligned} I_-(t) &= \frac{p/I(0)}{1 + [p/I(0) - 1]\, e^{-2pt}} \\ &\quad \cdot \{I_-(0)\cos 2\phi\, t + y(0)\sin 2\phi\, t\} \\ &\to \frac{p}{I(0)}\{I_-(0)\cos 2\phi\, t + y(0)\sin 2\phi\, t\}, \quad (12\,b) \end{aligned}$$

$$\begin{aligned} y(t) &= \frac{p/I(0)}{1 + [p/I(0) - 1]\, e^{-2pt}} \\ &\quad \cdot \{y(0)\cos 2\phi\, t - I_-(0)\sin 2\phi\, t\} \\ &\to \frac{p}{I(0)}\{y(0)\cos 2\phi\, t - I_-(0)\sin 2\phi\, t\}, \quad (12\,c) \end{aligned}$$

$$z(t) = z(0)\frac{p/I(0)}{1 + [p/I(0) - 1]\, e^{-2pt}} \to \frac{p}{I(0)}z(0), \qquad (12\,d)$$

with the constraint (7) for the four real initial values. When $\Delta\beta$ becomes nonzero, all solutions tend to one asymptotic state, independent of the initial values.

For an *analytical* discussion of the time dependent solution and of the phase transition from this solution to the asymmetric output steady state we introduce a transformation that is also suggested from the relation (10),

$$I_- = I\cos\psi, \quad y = I\sin\psi. \qquad (13)$$

Then

$$\dot\psi = -2\phi - \Delta\beta\, I\sin(2\psi), \qquad (14\,a)$$

$$\dot I = 2(p - I)\, I + 2\Delta\beta\, I^2\cos^2\psi. \qquad (14\,b)$$

Evidently for $|\Delta\beta\, I/(2\phi)| < 1$ a steady state solution for ψ is impossible. Therefore a time dependent solution emerges. But from (9), the larger $\Delta\beta$ the larger I can be. So there will be a critical value $\Delta\beta_c$ above which a stable steady state is possible again. This is the mechanism of the phase transition from the time dependent solution back again to a stable steady state in which

ψ^0 must be such that asymmetry with $I_-^0 \neq 0$ should be implied. If T denotes the period of I, i.e. $I(t + T) = I(t)$, we learn from (14) that $\psi(t + T) = \psi(t) - \mathrm{sign}(\phi)\,\pi$, so I_- and y have the period $2\,T$. T will be determined as a function of the parameters below. The magnitude of the intensity in the limit cycle state can be roughly estimated from (9) to vary between

$$p < I(t) < \frac{p}{1 - \Delta\beta_c}. \qquad (15)$$

One should note that the limit cycle solution is a singular one in the sense that for $\Delta\beta \neq 0$ the $\phi = 0$ solution and that of $\phi \to 0$ cannot be matched smoothly.

3. Small Interaction Asymmetry $\Delta\beta$

We now determine perturbatively the period and the intensity amplitude of the limit cycle laser output as functions of the parameters p, ϕ, and $\Delta\beta$. We proceed to transform (14) into a more compact form (without loss of generality assuming $\phi > 0$).

$$\frac{d\tilde\psi}{d\tau} = -2(1 + \mu\, r\sin\tilde\psi), \qquad (16\,a)$$

$$\frac{dr}{d\tau} = 2\tilde p\,(1 - r)\, r + \mu\, r^2\cos\tilde\psi, \qquad (16\,b)$$

where

$$\tilde\psi = 2\psi, \quad \tau = 2\phi\, t, \quad r = \frac{I}{\tilde I}, \qquad (16\,c)$$

$$\tilde I = \frac{p}{1 - \frac{1}{2}\Delta\beta}, \quad \mu = \tilde I\frac{\Delta\beta}{2\phi}, \quad \tilde p = \frac{p}{2\phi}. \qquad (16\,d)$$

We can consider μ and $\tilde p$ as two independent parameters.

In the parameter domain (8) of the time dependent solution we have

$$0 < \mu < \frac{\Delta\beta_c}{2 - \Delta\beta_c}\frac{p}{\phi} \equiv \mu_c, \qquad \text{and then}$$

$$\mu_c = \frac{1}{\sqrt{1 + (\phi/p)^2}} < 1, \qquad 2\tilde p > \mu_c. \qquad (17)$$

Thus μ can be taken as a small quantity and $\mu \to 0$ only when $\Delta\beta \to 0$. Naturally we now seek an expansion for the solution in terms of this small quantity,

$$\tilde\psi(\tau) = \sum_{n=0}^{\infty} \tilde\psi_n(\tau)\,\mu^n, \quad r(\tau) = \sum_{n=0}^{\infty} r_n(\tau)\,\mu^n, \qquad (18)$$

with a frequency-renormalized time variable $\bar{\tau}$ defined as

$$\tau = \left(1 + \sum_{n=1}^{\infty} c_n \mu^n\right) \bar{\tau}, \tag{19}$$

where the c_n are constants and will be chosen to compensate the secular terms in the perturbation equations. They are determined together with $\{\tilde{\psi}_n(\bar{\tau}), r_n(\bar{\tau})\}$ order by order in a self-consistent way [11]. Inversely we have

$$\bar{\tau} = \tau \left\{1 - c_1 \mu - (c_2 - c_1^2) \mu^2 + \cdots\right\}. \tag{20}$$

Noting that $r_n(\bar{\tau})$ must be periodic and, without loss of generality assuming $\tilde{\psi}(\bar{\tau}=0)=0$, which implies $\tilde{\psi}_n(\bar{\tau}=0)=0$ for all n, we have now

$$\frac{d\tilde{\psi}_0}{d\bar{\tau}} = -2, \quad \frac{dr_0}{d\bar{\tau}} = 2\tilde{p}(1-r_0)r_0, \tag{21}$$

which give the asymptotic solution

$$\tilde{\psi}_0(\bar{\tau}) = -2\bar{\tau}, \quad r_0(\bar{\tau}) = 1. \tag{22}$$

To the first order in μ we have

$$\frac{d\tilde{\psi}_1}{d\bar{\tau}} = -2(c_1 - \sin 2\bar{\tau}), \quad \frac{dr_1}{d\bar{\tau}} = -2\tilde{p}\,r_1 + \cos 2\bar{\tau}. \tag{23}$$

There is no secular term to be compensated with a nonzero c_1. The self-consistent solution to this order is

$$c_1 = 0, \quad \tilde{\psi}_1(\bar{\tau}) = 1 - \cos 2\bar{\tau},$$

$$r_1(\bar{\tau}) = \frac{1}{2(1+\tilde{p}^2)}(\sin 2\bar{\tau} + \tilde{p}\cos 2\bar{\tau}). \tag{24}$$

Up to the second order in μ we have

$$\frac{d\tilde{\psi}_2}{d\bar{\tau}} = -2\left(c_2 - \frac{3+2\tilde{p}^2}{4(1+\tilde{p}^2)}\right) - 2\cos 2\bar{\tau}$$
$$+ \frac{\tilde{p}}{2(1+\tilde{p}^2)}\sin 4\bar{\tau} + \frac{1+2\tilde{p}^2}{2(1+\tilde{p}^2)}\cos 4\bar{\tau}, \tag{25a}$$

$$\frac{dr_2}{d\bar{\tau}} = -2\tilde{p}\,r_2 + \frac{\tilde{p}}{4(1+\tilde{p}^2)} - \sin 2\bar{\tau} \tag{25b}$$
$$+ \frac{1}{2}\left(1 + \frac{1}{(1+\tilde{p}^2)^2}\right)\sin 4\bar{\tau} + \frac{\tilde{p}(3+\tilde{p}^2)}{4(1+\tilde{p}^2)^2}\cos 4\bar{\tau},$$

with the self-consistent solution

$$c_2 = \frac{3+2\tilde{p}^2}{4(1+\tilde{p}^2)}, \tag{26a}$$

$$\tilde{\psi}_2(\bar{\tau}) = -\sin 2\bar{\tau} + \frac{1+2\tilde{p}^2}{8(1+\tilde{p}^2)}\sin 4\bar{\tau}$$
$$+ \frac{\tilde{p}}{8(1+\tilde{p}^2)}(1 - \cos 4\bar{\tau}), \tag{26b}$$

$$r_2(\bar{\tau}) = \frac{1}{8(1+\tilde{p}^2)} - \frac{1}{2(1+\tilde{p}^2)}(\tilde{p}\sin 2\bar{\tau} - \cos 2\bar{\tau})$$
$$+ \frac{\tilde{p}}{4}\left(\frac{1}{4+\tilde{p}^2} + \frac{1}{(1+\tilde{p}^2)^2}\right)\sin 4\bar{\tau} \tag{26c}$$
$$- \frac{1}{8(1+\tilde{p}^2)^2}\left(\frac{8-7\tilde{p}^2}{4+\tilde{p}^2} + 3\tilde{p}^2\right)\cos 4\bar{\tau}.$$

In this order of approximation one gets from (20)

$$\bar{\tau} \approx \left(1 - \frac{3+2\tilde{p}^2}{4(1+\tilde{p}^2)}\mu^2\right)\tau. \tag{27}$$

Recalling that $\tau = 2\phi t$ and from the expression for $r_1(\bar{\tau})$ in (24) we have the renormalized frequency for the total intensity $I(t)$,

$$\nu \approx 2\phi \cdot 2 \cdot \left(1 - \frac{3+2\tilde{p}^2}{4(1+\tilde{p}^2)}\mu^2\right), \tag{28}$$

and the corresponding period, expressed in terms of the original parameters,

$$T = \frac{2\pi}{\nu} \approx \frac{\pi}{2\phi}\left\{1 + \frac{2(p^2+6\phi^2)}{p^2+4\phi^2}\left(\frac{p}{\phi}\frac{\Delta\beta}{2-\Delta\beta}\right)^2\right\}. \tag{29a}$$

$$T - T_0 \propto (\Delta\beta)^2 \quad \text{when} \quad \Delta\beta \to 0, \quad T_0 = \frac{\pi}{2\phi}. \tag{29b}$$

For $\Delta\beta \ll \Delta\beta_c$ and $|\phi/p| \ll 1$ it suffices

$$T \approx T_0\left\{1 + \frac{1}{2}\left(\frac{p\Delta\beta}{\phi}\right)^2\right\}. \tag{30}$$

Note that there is always a natural period T_0 due to the phase anisotropy. If the interaction asymmetry $\Delta\beta$ is switched on, the period increases by a factor that is quadratical in all parameters, $\sim \Delta\beta^2$, also $\sim p^2$, and $\sim \phi^{-2}$. A natural measure for the range of validity of (30) seems to be $p\,\Delta\beta/(2\phi) \lesssim 1$. Indeed, this suggestion will be confirmed later.

To the first order in μ the intensities are given by

$$I(t) = \frac{p}{1-\frac{1}{2}\Delta\beta}\left\{1 + \frac{\mu}{2(1+\tilde{p}^2)}(\sin 4\phi t + \tilde{p}\cos 4\phi t)\right\}, \tag{31a}$$

$$I_-(t) = I(t)\cos\left\{2\phi t - \frac{1}{2}\mu(1 - \cos 4\phi t)\right\}. \tag{31b}$$

From this explicit perturbative solution we easily determine the intensity bounds of the laser output in the limit cycle state

$$I_{\min, \max} = \frac{p}{1 - \frac{1}{2}\Delta\beta} \left(1 \mp \frac{\mu}{2\sqrt{1+\tilde{p}^2}} \right), \qquad (32\,\text{a})$$

$$|I_-|_{\max} \approx \frac{p}{1 - \frac{1}{2}\Delta\beta} \left(1 + \frac{\mu\,\tilde{p}}{2(1+\tilde{p}^2)} \right). \qquad (32\,\text{b})$$

Note that

$$I_{\min} < |I_-|_{\max} < I_{\max}. \qquad (33)$$

For $\Delta\beta \ll \Delta\beta_c$ and $|\phi/p| \ll 1$ and in lowest order of $\Delta\beta$,

$$I_{\min, \max} \approx p \begin{cases} 1, \\ 1 + \Delta\beta, \end{cases} \quad |I_-|_{\max} \approx p(1+\Delta\beta). \qquad (34)$$

Under the same parameter conditions the laser intensity oscillates through the limit cycle, which is centered at $I \approx p(1+\Delta\beta/2)$, with the amplitude span $I_{\max} - I_{\min} \approx p\,\Delta\beta$. The partial intensities periodically vary between 0 and $I_{\max}$, i.e. $p(1+\Delta\beta)$.

Numerical calculations confirm all these analytical results, especially the period correction (30) and the amplitude ranges (32).

A comparison with the $\mu = 0$ case (cf. (12)) is helpful to get a clear picture of the perturbed (anharmonic) periodic solution.

The total electric field can be approximately expressed as

$$E(t) \propto \sqrt{I} \left(\left| \cos\frac{\psi}{2} \right| e_+ - i \left| \sin\frac{\psi}{2} \right| e^{-2i\theta_2} e_- \right) \qquad (35)$$

$$\approx \sqrt{\frac{2p}{(2-\Delta\beta)}}$$

$$\cdot \left\{ 1 + \frac{\phi\,\mu}{2(p^2+4\,\phi^2)}(2\,\phi\sin 4\,\phi\, t + p\cos 4\,\phi\, t) \right\}$$

$$\cdot \left\{ \left| \cos\left[\phi\, t - \frac{1}{4}\mu(1-\cos 4\,\phi\, t) \right] \right| e_+ \qquad (36)$$

$$- i \left| \sin\left[\phi\, t - \frac{1}{4}\mu(1-\cos 4\,\phi\, t) \right] \right| e^{-2i\theta_2} e_- \right\}.$$

This describes a field of periodically varying polarization. In general it is elliptically polarized, including the limiting cases of linear and circular polarization.

4. Large Pumping p

The above perturbation approach is based on the small parameter $\mu = p\,\Delta\beta/[\phi(2-\Delta\beta)]$, which is of

purely mathematical origin. The perturbation series seems to converge fast enough. It actually is a Fourier series except for the renormalization process of the basic frequency. Higher order terms can be obtained straightforwardly. In practice the expansion is convenient only if μ is small enough, so that a few low order terms are sufficient to give a good approximation of the solution, and the time dependence is still nearly harmonic. But note that μ_c can reach the order of 1 if $|\phi/p|$ is small. Also it is difficult to confirm the exit point μ_c of the limit cycle range given by (17) from this perturbation approach (a rough approximation consists in letting the obtained perturbative frequency v be equal to zero and to solve from this equation for μ_c).

For an actual laser in operation we usually have $p \gg \phi(>0)$. This gives us a small parameter $\lambda = \phi/p$ of practical importance. With this small parameter we now can consider the behaviour of the laser for μ near its critical value μ_c from both sides, i.e. in the whole transition region, in a unified way. Important features will be the anharmonic oscillations of the intensity and of the electric field polarization, and the slowing down of the oscillation frequency towards the transition point.

In order to perform the perturbation expansion in terms of the new small parameter λ, we rewrite (16a, 16b) as

$$\frac{\mathrm{d}\tilde{\psi}}{\mathrm{d}\tau} = -2(1 + \mu\, r \sin\tilde{\psi}), \qquad (37\,\text{a})$$

$$\frac{\mathrm{d}r}{\mathrm{d}\tau} = \frac{1}{\lambda}(1-r)\, r + \mu\, r^2 \cos\tilde{\psi} \qquad (37\,\text{b})$$

and take μ and λ as independent parameters. It is obvious that the time scales for $\tilde{\psi}$ and r are of the order 1 and λ, respectively. This suggests to introduce the following frequency-renormalized perturbation series with two different time scales [11, 12]:

$$\tilde{\psi}(\tau) = \sum_{n=0}^{\infty} \tilde{\psi}_n(\bar{\tau})\, \lambda^n, \quad r(\tau) = \sum_{n=0}^{\infty} r_n(\hat{t})\, \lambda^n, \qquad (38)$$

where $\bar{\tau}$ is a new frequency-renormalized time variable and $\hat{t} = \bar{\tau}/\lambda$ characterizes the fast variations of $r(\tau)$,

$$\tau = \left(1 + \sum_{n=1}^{\infty} d_n\, \lambda^n \right) \bar{\tau}. \qquad (39)$$

The $\tilde{\psi}_n$ and r_n of this λ-expansion should not be confused with those of the μ-expansion in the previous section.

Although the critical value μ_c is already known from (17), it can also be determined in the frame of this

λ-perturbation approach. To do so, we introduce the expansion

$$\mu_c = \sum_{n=0}^{\infty} \mu_{cn} \lambda^n \qquad (40)$$

and express μ as

$$\mu = \tilde{\mu} + \sum_{n=1}^{\infty} \mu_{cn} \lambda^n = \tilde{\mu} + \mu_c - \mu_{c0}, \qquad (41)$$

where $\tilde{\mu}$ is assumed to be of the order of 1 or, more strictly, $\lambda \ll \tilde{\mu} \ll 1/\lambda$. In the limit cycle range $\mu < \mu_c$, so $\tilde{\mu} < \mu_{c0}$, while in the asymmetric steady state range $\mu \geq \mu_c$, so $\tilde{\mu} \geq \mu_{c0}$.

As before, we look for asymptotically stable solutions order by order and choose the initial condition $\tilde{\psi}(\tau = 0) = 0$ in the case of the time dependent solutions.

In lowest order of λ we have

$$\frac{d\tilde{\psi}_0}{d\bar{\tau}} = -2\{1 + \tilde{\mu} r_0(\hat{t}) \sin \tilde{\psi}_0\}, \qquad (42a)$$

$$\frac{dr_0}{d\hat{t}} = (1 - r_0) r_0, \qquad (42b)$$

which gives us the following asymptotic solution together with the value of μ_{c0},

$$\mu_{c0} = 1, \quad r_0 = 1,$$

$$\tilde{\psi}_0 = \begin{cases} -2 \arctan(\tilde{\mu} - \sqrt{\tilde{\mu}^2 - \mu_{c0}^2}), & \tilde{\mu} \geq \mu_{c0}; \\ -2 \operatorname{Arctan}\left\{\tilde{\mu} + \sqrt{\mu_{c0}^2 - \tilde{\mu}^2} \tan\left(\sqrt{\mu_{c0}^2 - \tilde{\mu}^2}\, \bar{\tau} - \arctan\frac{\tilde{\mu}}{\sqrt{\mu_{c0}^2 - \tilde{\mu}^2}}\right)\right\}, & \tilde{\mu} < \mu_{c0}. \end{cases}$$

$$(43a)$$
$$(43b)$$

This represents analytically a transition of Hopf bifurcation type. Note the unusual time dependence of $\tilde{\psi}_0(\bar{\tau})$. It describes highly anharmonic oscillations with quite a long period near the transition point. By the way, $\mu_{c0} = 1$ confirms our earlier finding that $p\Delta\beta/(2\phi) \lesssim 1$ characterizes the relevant range of the limit cycle regime, cf. the comments after (30).

From its definition in (41) we have now

$$\tilde{\mu} = \mu - \mu_c + 1. \qquad (44)$$

To solve the higher order equations we proceed separately with the steady state and with the time dependent state. The n-th order transition point correction μ_{cn} will be determined in such a way that the n-th order steady state solution is regular (remember that the asymmetric output steady state does not exist when $\Delta\beta < \Delta\beta_c$, i.e., $\tilde{\mu} < \mu_{c0}$).

The first order equations for the steady state in the range $\tilde{\mu} \geq \mu_{c0}$,

$$r_1 \tilde{\mu} \sin \tilde{\psi}_0 + \tilde{\psi}_1 \tilde{\mu} \cos \tilde{\psi}_0 + \mu_{c1} \sin \tilde{\psi}_0 = 0, \quad (45a)$$

$$r_1 - \tilde{\mu} \cos \tilde{\psi}_0 = 0, \qquad (45b)$$

have the solutions

$$\mu_{c1} = 0, \quad \tilde{\psi}_1 = -\tilde{\mu} \sin \tilde{\psi}_0 = 1,$$
$$r_1 = \tilde{\mu} \cos \tilde{\psi}_0 = \sqrt{\tilde{\mu}^2 - \mu_{c0}^2}. \qquad (46)$$

Higher order terms can be evaluated routinely. Note that the expansion of the exact critical value from (17),

$$\mu_c = \frac{1}{\sqrt{1+\lambda^2}} = 1 - \frac{1}{2}\lambda^2 + \cdots, \qquad (47)$$

shows agreement with our findings of $\mu_{c0} = 1$ and $\mu_{c1} = 0$.

With $\mu_{c1} = 0$ in mind, we derive the first order equations for the limit cycle state in the parameter range $\tilde{\mu} < \mu_{c0}$,

$$\frac{d\tilde{\psi}_1}{d\bar{\tau}} = -2\{\tilde{\psi}_1 \tilde{\mu} \cos \tilde{\psi}_0(\tau) + r_1(\hat{t}) \tilde{\mu} \sin \tilde{\psi}_0(\tau) + d_1[1 + \tilde{\mu} \sin \tilde{\psi}_0(\tau)]\}, \qquad (48a)$$

$$\frac{dr_1}{d\hat{t}} = -r_1 + \tilde{\mu} \cos \tilde{\psi}_0(\lambda \hat{t}). \qquad (48b)$$

The r_1-equation (48b) can be simply solved using an adiabatic approximation to get

$$r_1(\hat{t}) \approx \tilde{\mu} \cos \tilde{\psi}_0(\lambda \hat{t}). \qquad (49a)$$

This is reasonable on the time scales of the homogeneous and inhomogeneous terms of (48b) because $\lambda \ll 1$. We then have the self-consistent solutions for d_1 and for $\tilde{\psi}_1(\bar{\tau})$,

$$d_1 = 0, \qquad (49b)$$

$$\tilde{\psi}_1(\bar{\tau}) = \{1 + \tilde{\mu} \sin \tilde{\psi}_0(\tau)\} \ln\{1 + \tilde{\mu} \sin \tilde{\psi}_0(\tau)\} - \tilde{\mu} \sin \tilde{\psi}_0(\tau). \qquad (49c)$$

d_1 vanishes due to the fact that there is no secular term in $\tilde{\psi}_1(\bar{\tau})$.

From its definition in (41) it is assumed that $\lambda \ll \tilde{\mu} \ll 1/\lambda$. Now, knowing $\mu_{c1} = 0$, we can safely relax this restriction to $\lambda^2 \ll \tilde{\mu} \ll 1/\lambda$.

To conclude, a good approximation for the limit cycle solution with $\lambda^2 \ll \mu < \mu_c$ can be expressed as follows:

$$\tilde{\psi}(\tau) \approx -2\,\mathrm{Arctan}\left\{\mu + \sqrt{\mu_c^2 - \mu^2}\,\tan\left(\sqrt{\mu_c^2 - \mu^2}\,\tau - \arctan\frac{\mu}{\sqrt{\mu_c^2 - \mu^2}}\right)\right\}$$

$$\approx -2\,\mathrm{Arctan}\frac{\tan\sqrt{\mu_c^2 - \mu^2}\,\tau}{\sqrt{\mu_c^2 - \mu^2} + \mu\tan\sqrt{\mu_c^2 - \mu^2}\,\tau}; \tag{50a}$$

$$r(\tau) \approx 1 + \lambda\,\mu\cos\tilde{\psi}(\tau) \approx 1 + \lambda\,\mu\left\{1 - \frac{1 - \cos 2\sqrt{\mu_c^2 - \mu^2}\,\tau}{1 - \mu^2\cos 2\sqrt{\mu_c^2 - \mu^2}\,\tau + \mu\sqrt{\mu_c^2 - \mu^2}\sin 2\sqrt{\mu_c^2 - \mu^2}\,\tau}\right\}. \tag{50b}$$

We have, in some places, applied the approximations $\mu_c = \mu_{c0} + O(\lambda^2) \approx \mu_{c0} = 1$ and $\tilde{\mu} = \mu + O(\lambda^2) \approx \mu$. For the original variables we have

$$I(t) = \frac{p}{1 - \frac{1}{2}\Delta\beta}\{1 + \lambda\,\mu\cos\tilde{\psi}(2\phi t)\}, \tag{51a}$$

$$I_-(t) \approx I(t)\cos\left\{\frac{1}{2}\tilde{\psi}(2\phi t)\right\}. \tag{51b}$$

The corresponding period can be identified as

$$T \approx \frac{T_0}{\sqrt{\mu_c^2 - \mu^2}}, \quad T_0 = \frac{\pi}{2\phi}. \tag{52a}$$

$$T \propto (\Delta\beta_c - \Delta\beta)^{-1/2}, \quad \text{when} \quad \Delta\beta \nearrow \Delta\beta_c. \tag{52b}$$

It clearly exhibits slowing down of the oscillations towards the upper transition point $\Delta\beta_c$. The scaling expressed by (52b) also holds beyond small λ. In fact from (43), (39), (16c), and (44)

$$T = \frac{T_0}{\sqrt{2(\mu_c - \mu) - (\mu_c - \mu)^2}}\left(1 + \sum_{n=1}^{\infty} d_n\lambda^n\right). \tag{53}$$

According to (39), the quantity in the brackets relates the two physical times τ and $\bar{\tau}$, and therefore can neither be zero nor singular in the limit $\mu \nearrow \mu_c$. Thus the scaling is in effect only determined by the denominator in (53) and agrees with (52a).

The lower and upper bounds for the intensities are as follows:

$$I_{\min,\max} = \frac{p}{1 - \frac{1}{2}\Delta\beta}(1 \mp \lambda\tilde{\mu}), \quad |I_-|_{\max} = I_{\max}. \tag{54}$$

Near the transition point they are

$$I_{\min,\max} \approx \begin{cases} p, \\ p + 2\phi, \end{cases} \quad |I_-|_{\max} \approx p + 2\phi. \tag{55}$$

So the laser intensity oscillates through the limit cycle, which is centered at $I \approx p + \phi$, with the span $I_{\max} - I_{\min} = 2\phi$. The partial intensities periodically vary between 0 and $I_{\max}$. Note that all these oscillations are now highly anharmonic.

Figures 3 and 4 demonstrate our analytical results (51). We recognize nice agreement with the results of the numerical integration of (3)–(5) in Figs. 1 and 2.

The electric field can be directly obtained from (35), and the comments at the end of Sect. 3 are valid here too.
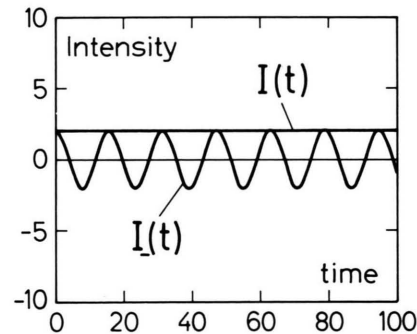

Fig. 3. The analytical solutions (51) from the perturbation expansion for the intensities. The parameters used are the same as in Figure 1. Compare the nice agreement with Fig. 1 after transients.
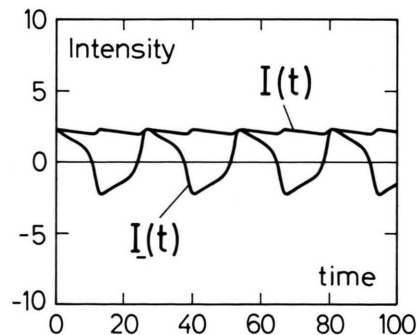

Fig. 4. The same as in Fig. 3 but with the parameters used in Figure 2. The agreement after transients is also good.

It should be pointed out that the two perturbation approaches coincide quite well in the common parameter region $\lambda^2 < \mu \ll 1$ as expected. In fact we can obtain almost the same as in Fig. 3 from the analytical expressions (31).

5. Concluding Remarks

We have considered in detail the limit cycle solution of the polarization dynamics for the two electric field polarization modes of a monochromatic gas laser with an intensity coupling asymmetry $\Delta\beta$ and a linear phase anisotropy ϕ in the cavity loss. This solution results from mode cooperation as well as competition and only shows up in a parameter window $0 < \Delta\beta < \Delta\beta_c$ of the lasing state $p > 0$. The upper critical value $\Delta\beta_c$ crucially depends on the linear phase anisotropy, and the window shrinks when the pump rate becomes larger. This solution is intermediate between the symmetric output steady state with linear polarization and the asymmetric output steady state with almost circular polarization. In this parameter window both the intensity and the polarization oscillate periodically and, while the total intensity varies periodically with a small change of magnitude, the two partial intensities vary between zero and their maximum with a period twice as large as that of the total intensity oscillation. If the intensity coupling asymmetry is small, all these oscillations are nearly harmonic, but they become highly anharmonic and with longer period when the intensity coupling asymmetry approaches the upper critical value $\Delta\beta_c$. This corresponds to two different Hopf bifurcations when entering the window at $\Delta\beta = 0$ or at $\Delta\beta = \Delta\beta_c$.

Using frequency-renormalized perturbation methods we have considered two limiting cases of small intensity coupling asymmetry and large pumping, respectively. In the former case it actually is a Fourier series expansion with a renormalization of the basic frequency. The result is a good approximation only for the range of nearly harmonic time dependence. In the latter case there clearly exist two different time scales. It demands a more elaborate perturbation technique and, at the same time, allows to apply an adiabatic approximation in solving the perturbation equations. The strongly anharmonic time dependence can thus well be studied.

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